Pseudorandom Approximate Unitary Designs

Or one way to sample uniformly random quantum circuits

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PCMI Research Talk

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Joint work with:

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CMU

Rocco Servedio
Columbia University

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Question:

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Note that "morally" we need about $\widetilde{\Theta}(N^2)$ bits of entropy

Consider $\mathrm{Sym}(N)$ — the group of $N \times N$ permutation matrices

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A distribution ν on $\mathrm{U}(N)$ "efficiently" samplable with $\ll N$ bits of entropy, such that:

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Definition (ε-approximate unitary k-design):

$$\left\| \mathop{\mathbf{E}}_{\boldsymbol{X} \sim \nu} \left[\boldsymbol{X}^{\otimes k} \otimes (\overline{\boldsymbol{X}})^{\otimes k} \right] - \mathop{\mathbf{E}}_{\boldsymbol{X} \sim \mathrm{U}(N)} \left[\boldsymbol{X}^{\otimes k} \otimes (\overline{\boldsymbol{X}})^{\otimes k} \right] \right\|_1 \leqslant \varepsilon$$

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Intuition: $\boldsymbol{X}^{\otimes k} \otimes (\overline{\boldsymbol{X}})^{\otimes k}$ is a $N^{2k} \times N^{2k}$ matrix
entries are products of k entries of $\boldsymbol{X}$ and their conjugates

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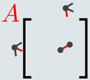

Random walk step $\equiv$ Picking a permutation uniformly at random

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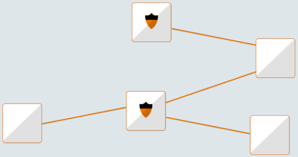
Fact

A $(N, \varepsilon/N^k, k)$-TPE is an ε-approximate unitary k-design

Part I: Motivation



• What is known? •

$N = 2^n$

Method	Bits of Entropy	Efficient?
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(Note: our work also achieves designs for other groups, like $\mathrm{O}(N)$)

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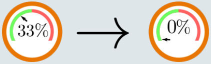
Note: our work also has some classical applications

Let's look at the motivation behind them

• Classical Detour: Why Pseudorandom? •

1 Convert random algorithms to deterministic using similar time

Ex: Primes in P, Undirected Reachability



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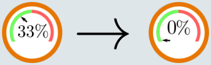
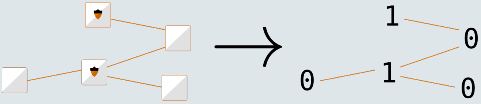


② Construct explicit objects whose existence is only guaranteed by the probabilistic method

Ex: Expanders, Efficient Codes



Definition: Pseudorandom Generator G

$G : \{0,1\}^t \to \{0,1\}^n$ ε-fools a family of tests $\mathcal{F}$, where $f \in \mathcal{F}$ is $f : \{0,1\}^n \to \{0,1\}$ if

$$\forall f \in F, \qquad \left|\boldsymbol{P}_{x \sim U_n}[f(x) = 1] - \boldsymbol{P}_{z \sim U_t}[f(G(z)) = 1]\right| \leqslant \varepsilon$$

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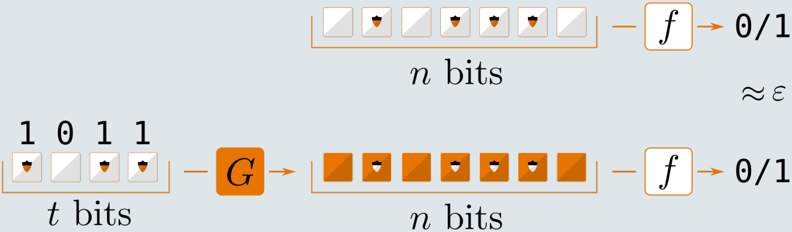
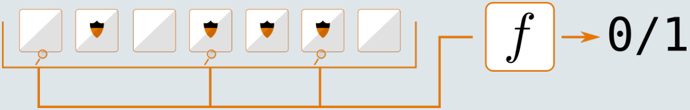


• k-wise independence •

Consider the family of k-wise independent tests:

$$f \in \mathcal{F} \text{ only looks at most } k \text{ bits of the input}$$



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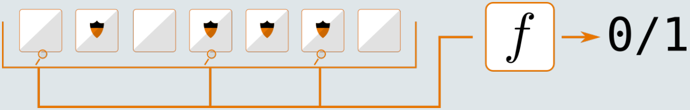

Example: k-wise uniform bits

G is k-wise independent if for $\boldsymbol{x} \sim U_N$ and all distinct $i_1, i_2, \ldots i_k$

i_1th bit of $G(\boldsymbol{x})$, ..., i_kth bit of $G(\boldsymbol{x})$ are uniform

i.e. the probability of seeing any length k binary string is $1/2^k$

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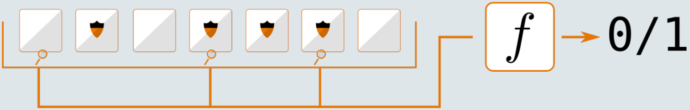

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Theorem [ABI'85]

Such a G exists with $t = O(kn)$

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$[N]_k \to k$ distinct from $1 \dots N$

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$\Pi \subseteq S_N$ is k-independent if for $\pi \in \Pi$

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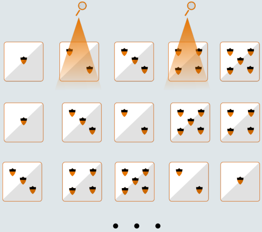

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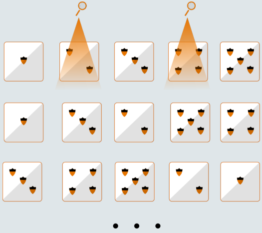

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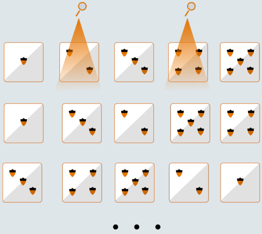

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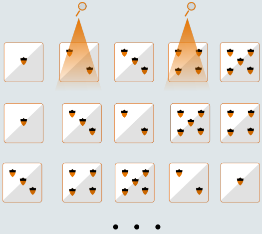

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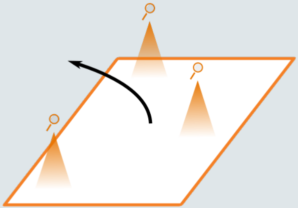
Many applications, e.g. cryptography, coding theory, expanders ...

Part II: General Framework



• Our framework •

A Baby Distribution

1. Construct $\mathcal{M}$, a set of matrices in $\mathrm{U}(N)$, such that:

- $$\left\| \mathop{\mathbf{E}}_{\mathcal{M}} \left[\boldsymbol{M}^{\otimes k,k} \right] - \mathop{\mathbf{E}}_{\mathrm{U}(N)} \left[\boldsymbol{M}^{\otimes k,k} \right] \right\|_{\mathrm{op}} \leqslant 1 - \frac{1}{\mathrm{poly}(k) n}$$
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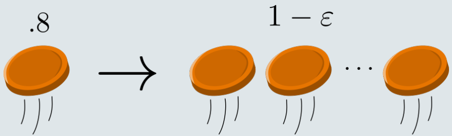
Error Reduction

2. Use $\mathcal{M}$ to obtain $\hat{\mathcal{M}}$, such that:

- $$\left\| \mathop{\mathbf{E}}_{\hat{\mathcal{M}}} \left[M^{\otimes k,k} \right] - \mathop{\mathbf{E}}_{\mathrm{O}(N)} \left[M^{\otimes k,k} \right] \right\|_{\mathrm{op}} \leqslant \delta$$
- $|\hat{\mathcal{M}}|$ small $\to O(kn + \log(1/\delta))$ bits of entropy

Part III: Error Reduction

• Intuition •



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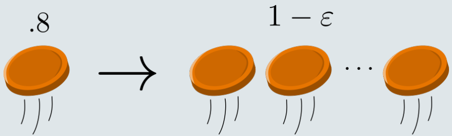

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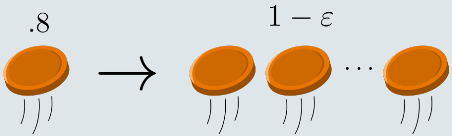

$$\hat{\mathcal{M}} = \mathcal{M}^t, \text{ where } \mathcal{M}^t = \{M_1 \cdot M_2 \cdot \ldots \cdot M_t | M_i \in \mathcal{M}\}$$

• Some Facts About Tensors •

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$$\mathbf{E}_{\mathrm{U}(N)} \left[\boldsymbol{M}^{\otimes k,k} \right]^2 = \mathbf{E}_{\mathrm{U}(N)} \left[\boldsymbol{M}^{\otimes k,k} \right],$$

so it's a projector matrix and $\Pi_{\mathrm{U}(N)} := \mathbf{E}_{\mathrm{U}(N)} \left[\boldsymbol{M}^{\otimes k,k} \right]$

Lemma: Error reduction

If $\hat{\mathcal{M}} = \mathcal{M}^t$ then:

$$\left\| \mathop{\mathbf{E}}_{\hat{\mathcal{M}}} \left[\boldsymbol{M}^{\otimes k,k} \right] - \mathop{\mathbf{E}}_{\mathrm{U}(N)} \left[\boldsymbol{M}^{\otimes k,k} \right] \right\|_{\mathrm{op}} \leqslant \varepsilon^t$$

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Proof. Assume $t = 2$ (general case follows from this).

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The given operator norm bound now gives the result $\square$

• A Decent Reduction... •

Using error reduction, pick $t = \mathrm{poly}(\log N, k)\log(1/\delta)$, we conclude:

- $$\left\| \mathop{\mathbf{E}}_{\hat{\mathcal{M}}}\left[\boldsymbol{M}^{\otimes k,k} \right] - \Pi_{\mathrm{U}(N)} \right\|_{\mathrm{op}} \leqslant \delta$$

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- But... $|\hat{\mathcal{M}}| = |\mathcal{M}|^t \to O(\mathrm{poly}(n,k) \log(1/\delta))$ bits of entropy

• Intuition for a Better Reduction •

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Label the vertices with matrices from $\mathcal{M}$, so $v \in V$ and $M_v \in \mathcal{M}$

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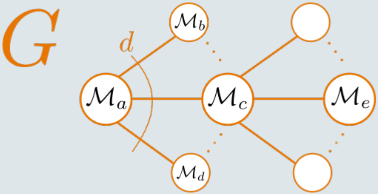

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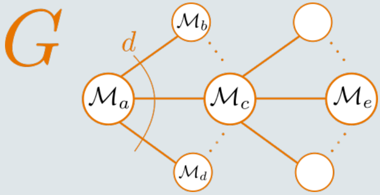

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Challenge 1: Prove that this reduces the error, like the previous reduction

Challenge 2: Pick appropriate expander graphs
(derandomized squaring [RTV'05] [RV'05])

• Technical Result •

Theorem: Operator Reduction

Let $\mathcal{M} = (M_1, \ldots, M_c)$ be a matrices in $\mathbb{R}^{r \times r}$ satisfying $\|M_i\|_{\mathrm{op}} \leqslant 1$ for all i and $\left\| \mathbf{E}_{\mathcal{M}} \left[\boldsymbol{M}^{\otimes k,k} \right] - \Pi \right\|_{\mathrm{op}} \leqslant 1 - \varepsilon$

There is a strongly explicit, space-minimal algorithm that outputs a sequence Q of $N' = O(c/(\varepsilon^{11.25}\delta^{10}))$ monomials over $M_1, \ldots, M_c$, each of length $L = 8\log_2(1/\delta)/\varepsilon^{1.25}$, such that:

$$\left\| \mathbf{E}_{\hat{\mathcal{M}}} \left[\boldsymbol{M}^{\otimes k,k} \right] - \Pi \right\|_{\mathrm{op}} \leqslant \delta, \text{ for } \hat{\mathcal{M}} = \mathcal{M}^Q$$

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Translation:

- $$\left\| \mathop{\mathbf{E}}_{\hat{\mathcal{M}}} \left[\boldsymbol{M}^{\otimes k,k} \right] - \Pi_{\mathrm{U}(N)} \right\|_{\mathrm{op}} \leqslant \delta$$
- $|\hat{\mathcal{M}}| \leqslant \mathrm{poly}(2^{nk}/\delta) \to O(kn + \log(1/\delta))$ bits of entropy

Part IV: A Baby Distribution

$$M_1 \otimes \mathbb{I} \otimes \mathbb{I} \otimes M_2$$

$$|1\rangle \quad |2\rangle \quad |3\rangle \quad |4\rangle$$

• Recall our Goal •

1. Construct $\mathcal{M}$, a set of matrices in $\mathrm{U}(N)$, such that:

- $$\left\| \mathop{\mathbf{E}}_{\mathcal{M}} \left[\boldsymbol{M}^{\otimes k,k} \right] - \Pi_{\mathrm{U}(N)} \right\|_{\mathrm{op}} \leqslant 1 - \frac{1}{\mathrm{poly}(k)n}$$
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Actually, this is given in [BHH'19], [HHJ'20], [Haf'22]!

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$N = 2^n$

Let $P \subset \mathrm{U}(2^\ell)$ be a finite set and $E \subseteq [n]_\ell$

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$$M \in \mathbf{U}(4) \qquad e = \{1, 4\}$$

$$M = M_1 \otimes M_2$$

$$M_e = M_1 \otimes \mathbb{I} \otimes \mathbb{I} \otimes M_2$$

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Theorem: Non-trivial gap construction

For a fixed small positive n_0, suppose P_{n_0} is a universal set in $\mathrm{U}(N)$

Then $\mathcal{M} = P_{n_0} \times \binom{[n]}{n_0}$ satisfies:

$$\left\| \mathop{\mathbf{E}}_{\mathcal{M}} \left[\boldsymbol{M}^{\otimes k, k} \right] - \Pi_{\mathrm{U}(N)} \right\|_{\mathrm{op}} \leqslant 1 - \frac{1}{\mathrm{poly}(k) n}$$

• Proof Outline •

Abuse notation: let $P_1 \lesssim \alpha P_2$ be

$$\left\| \mathop{\mathbf{E}}_{P_1} \left[\boldsymbol{M}^{\otimes k,k} \right] - \Pi_{\mathrm{U}(N)} \right\|_{\mathrm{op}} \leqslant \alpha \left\| \mathop{\mathbf{E}}_{P_2} \left[\boldsymbol{M}^{\otimes k,k} \right] - \Pi_{\mathrm{U}(N)} \right\|_{\mathrm{op}}$$

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P_{n_0} universal

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Then:

$$P_{n_0} \times \binom{[n]}{n_0} \lesssim \kappa_{n_0} \mathrm{U}(2^{n_0}) \times \binom{[n]}{n_0}$$

From [BdS16] and [BG12]

• Proof Outline •

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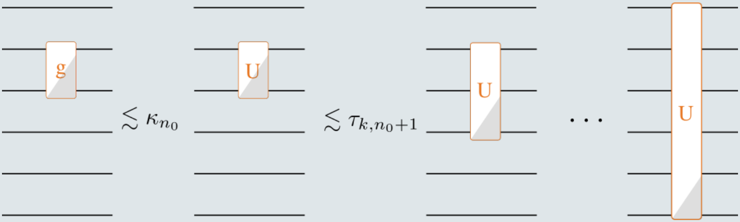
• Proof Visualization •



Thanks!